Domino Tilings of a Rectangular Chessboard

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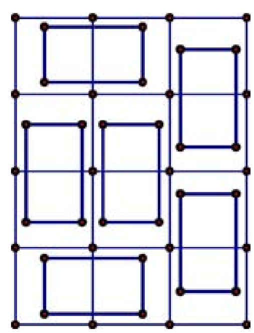
1 The Problem

Consider an M by N chessboard. Place $\frac{MN}{2}$ dominoes (2 by 1 rectangles) on the chessboard so that no dominoes overlap and no squares are uncovered. The following is a valid tiling of a 4 by 3 chessboard:



Let the number of tilings of an M by N chessboard be $Z(M,N)$. The goal of this class is to prove that

$$Z(M,N) = \prod_{m=1}^{M}\prod_{n=1}^{N}(4\cos^2\frac{m\pi}{M+1} + 4\cos^2\frac{n\pi}{N+1})^{\frac{1}{4}}$$

1.1 Problems

1. Find $Z(2,2), Z(3,2)$, and $Z(4,3)$. Verify the formula for $Z(M,N)$ in these cases.

2. What is $Z(M,N)$ if both M and N are odd?.

3. Find $Z(2,N)$.

4. Prove that $Z(3,2N) = 4Z(3,2N-2) - Z(3,2N-4)$ for $N \geq 2$ and $Z(3,0) = 1$.

Hint: First, prove that $Z(3,2N+2) = 3Z(3,2N) + 2\sum_{n=0}^{N-1} Z(3,2n)$.

2 Matrices, Permutations, and Determinants

A **permutation** is a function that changes the order of the numbers $\{1, 2, \cdots, n\}$ for some positive integer n. All permutations are sequences of **transpositions**, or functions that switch two numbers in a permutation. The **sign** of a permutation, denoted by $sgn(\sigma)$ for a permutation σ, is $(-1)^k$ where k is the number of transpositions needed to obtain σ from $\{1, 2, \cdots, n\}$. Note that the sequence of transpositions needed to obtain σ is not unique.

The **determinant** of a matrix is a function on square matrices that outputs a scalar. The determinant has the following properties:

Linearity: $\det(v_1, \cdots, au_1+bu_2, \cdots, v_n) = a\det(v_1, \cdots, u_1, \cdots, v_n)+b\det(v_1, \cdots, u_2, \cdots, v_n)$.

Alternation: $\det(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_n) = -\det(v_1, \cdots, v_{i-1}, v_j, v_{i+1}, \cdots, v_{j-1}, v_i, v_{j+1}, \cdots, v_n)$.

Identity: $\det(e_1, \cdots, e_n) = 1$, where e_i is a vector with a 1 in the i^{th} position and the rest of the elements 0.

2.1 Problems

1. Find the sign of the permutation σ that takes $\{1, 2, 3, 4, 5\}$ to $\{4, 3, 5, 1, 2\}$.

2. Find the determinant of

$$\begin{pmatrix} 1 & 2 & 2 \\ 3 & 5 & 7 \\ 6 & 1 & 5 \end{pmatrix}$$

.

3. What is the sign of the permutation that takes $\{1, 2, \cdots, k\}$ to $\{2, 3, \cdots, k, 1\}$?

4. Prove that the determinant is unique by proving that $\det(M) = \sum_{\tau \in S_n} sgn(\tau) M_{1\tau(1)} \cdots M_{n\tau(n)}$,

where S_n is the set of permutations of length n.

5. Prove that the sign function is well defined, i.e. that it has the same value for all transposition sequences.

Hint: Consider the polynomial $\prod_{i<j}(x_i - x_j)$.

6. Find the determinant of

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

.

where B is a square matrix and 0 is a square matrix of zeros with the same dimensions as B.

3 Graphs and Adjacency Matrices

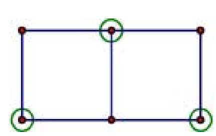
A **graph** is a set of points and edges. Label the vertices of the graph with numbers from 1 to n where n is the number of vertices in the graph. The **adjacency matrix** of a graph is a matrix M such that $M_{ij} = 1$ if vertices i and j are connected and 0 otherwise.

A graph is **bipartite** if its vertices can be divided into two groups so that no vertices within a group are connected. A **perfect matching** is a graph in which each vertex is connected to exactly one other vertex.

Notice that the domino tiling problem is equivalent to finding the number of perfect matchings of a grid of M by N points.

3.1 Problems

1. What is the adjacency matrix of the following graph:



What do you notice about the adjacency matrix of this graph?

2. Make a perfect matching with vertex sets $\{1, 2, 3, 4\}$ and $\{1', 2', 3', 4'\}$ and construct a matrix M with $M_{ij} = 1$ if vertices i and j' are connected and $M_{ij} = 0$ otherwise. Compute the determinant of this matrix, listing each term of the calculation. What does each term mean? Can this determinant be adjusted in some way to obtain something useful?

3. Make a weighted, adjusted adjacency matrix in which $M_{ij} = i$ ($i = \sqrt{-1}$) if vertices i and j' are connected vertically, 1 if i and j' are connected horizontally, and 0 otherwise. Compute the determinant of this weighted matrix. What information can you gather from this determinant?

4. Prove that you will always get the information found in Question 3. Ask me for hints.

4 Eigenvalues and Eigenvectors

For a matrix M, all vectors v such that $Mv = \lambda v$ for some scalar λ are called **eigenvectors**. All λ values are called **eigenvalues**.

Eigenvalues can be used to diagonalize a matrix. Let P be a square n by n matrix with n eigenvectors of M as the columns. Notice that $MP = PD$, where D is the matrix with the n corresponding eigenvalues on the diagonal and 0s everywhere else. If $\det P \neq 0$, then $M = PDP^{-1}$. This is called the **diagonalization** of M.

4.1 Problems

1. What are the eigenvalues of

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

?

Multiply this matrix by the vector $< 1, 0 >$ repeatedly. Do you notice a pattern?

2. Take exponents of the matrix in the previous exercise using the diagonalization of the matrix. What does this say about the pattern found in the previous exercise?

3. Prove that the determinant of a matrix is the product of its eigenvalues.

5 Cartesian Products of Graphs

For two graphs G and H, their **Cartesian product**, denoted $G\Box H$, is the graph with vertices that are ordered pairs (g, h) where (g_1, h_1) and (g_2, h_2) are adjacent iff $h_1 = h_2$ and g_1 is connected to g_2 in G or vice versa.

The **tensor product** of two matrices, denoted $A \otimes B$, is the matrix

$$\begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{pmatrix}$$

where each element is actually a m' by n' block of entries, where B is a m' by n' matrix and A is a m by n matrix.

Note that the chessboard graph is equivalent to the Cartesian product of a vertical row of M points with edge weighting i and a horizontal row of N points with edge weighting 1.

5.1 Problems

1. Find the Cartesian product of two triangles.

2. Prove that for matrices A, B, C, and D $AC \otimes BD = (A \otimes B)(C \otimes D)$ if multiplication makes sense.

3. Prove that the adjacency matrix of $G\Box H$ is $A_G \otimes I_h + I_g \otimes A_H$, where A_G is the adjacency matrix of G and g is the number of vertices in G.

6 Tying it all together

6.1 Problems

1. What are the eigenvalues of the adjacency matrix of a row graph (a $1 \times M$ or $1 \times N$ grid of points)? Hint: Use the following identity:

$$\sin((k-1)\theta) + \sin((k+1)\theta) = 2\cos\theta \sin k\theta.$$

2. What are the eigenvalues of the overall adjacency matrix of the chessboard graph?

3. Find the number of domino tilings of an $M \times N$ chessboard.